

Casimir Effect at finite temperature for the CPT-even extension of QED

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By the thermofield dynamics (TFD) formalism we obtain the energy-momentum tensor for the Electromagnetism with Lorentz Breaking Even term of the Standard Model Extended (SME) Sector in a topology $S^1 \times S^1 \times R^2$. We carry out the compactification by a generalized TFD-Bogoliubov transformation that is used to define a renormalized energy-momentum tensor, and the Casimir energy and pressure at finite temperature are then derived. A comparative analysis with the electromagnetic case is developed, and we remark the influence of the background in the traditional Casimir effect.

I. INTRODUCTION

The Standard Model of particle physics (SM) has gotten success in its final test: the discovery of the Higgs boson at the LHC in 2013. Despite the tremendous success of this model, there is a fundamental question to be answered, which is the justification for the Higgs mass. There is another problem related to the mass of the Higgs boson: as the energy grows beyond the SM energy scale, the radiative corrections makes the mass of the Higgs boson to diverge (the problem of the Hierarchy). Also there is a lack of any explanation for Dark Matter, and the unbalance between matter-antimatter. Also, recently, The Standard Model

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of particle physics (SM) has not gotten success in explaining the origin of electron's electric dipole moment (EDM), d_e , and its experimental upper bounds [1]. Theories beyond the SM predict a small, but potentially measurable EDM ($d_e \leq 10^{-29} e \cdot cm$) [2], which presents an asymmetric charge distribution along the spin axis. Therefore, with this experimental result, it is necessary to investigate the physics beyond the Standard Model. Despite the great success of SM to give an overview of the microscopic processes through a field theory that unifies the weak and electromagnetic interaction SM presents nowadays some limitations.

For view of these limitations of the SM, one is greatly motivated to propose models that can give us hints about a more fundamental theory. In 1989, in a string theory environment, Kostelecký and Samuel [3] realized an interesting possibility in order to establish the spontaneous violation of symmetry through non-scalar fields (vacuum of fields that have a tensor nature) based on a string field theory environment. A consistent description of fluctuations around this new vacuum is obtained if the components of the background field are constant, and by the fact that this new minimum be a background not scalar, the Lorentz symmetry is spontaneously broken [4]. This possibility of extending the Standard Model was made for fields that belong to a more fundamental theory in which, in turn, can be spontaneously violated based on a specific criterion. It is desirable that any extension of the model could keep the gauge invariance, the conservation of energy and momentum, and the covariance under observer rotations and boosts. This proposal was known as Standard Model Extension (SME) [5] .

It is well-known that the presence of terms that violate the Lorentz symmetry imposes at least one privileged direction in the spacetime. Nowadays, studies in relativistic quantum effects [6] that stem from a non minimal coupling with Lorentz symmetry breaking [7] has opened the possibility of investigating new implications in quantum mechanics that this violator background can promote.

An interesting question is investigate the influence of privileged directions from Spontaneous Lorentz Violations coupled with the zero modes of the electromagnetic field, i. e., in which way the vacuum fluctuation can be affected?

The Casimir effect is one of the most remarkable manifestations of vacuum fluctuations, and for the electromagnetic field, it consists in the attraction between two metallic plates, parallel each other, embedded into the vacuum [8]. The attraction is due to a fluctuation of the fundamental energy of the field caused by the presence of planes, which select the

electromagnetic vacuum modes by boundary conditions [9–12]. In general, the Casimir effect is then a modification in the vacuum energy of a given quantum field due to the imposition of boundary conditions or topological effects on this field. The measurements of this effect in great accuracy in the last decade has gained attention of the theoretical and experimental community [13, 14]. One practical implication of these achievements is the development of nanodispositives. [15–17].

In this paper we focus on the possibility of electromagnetism with breaking of Lorentz symmetry in such way that the CPT symmetry is preserved. The even sector of the SM with a vector decomposition is taken into account [18, 19], and we analyze the influence on such breaking of symmetry in the Casimir effect with thermal field treatment.

Considering that a thermal field theory is a quantum field compactified in a topology $S^1 \times R^3$, a result of the KMS (Kubo, Martin, Schwinger) condition, this apparatus has been used to describe field theories in toroidal topologies [20–25]. In terms of TFD, the Bogoliubov transformation has then been generalized to describe thermal and space-compactification effects with real (not imaginary) time. Here we consider a Bogoliubov transformation to take into account the Lorentz spontaneous violation in a topology $S^1 \times S^1 \times R^2$. Such a mechanism is quite suitable to treat, in particular, the Casimir effect. This is a consequence of the nature of the propagator that is written in two parts: one describes the flat (Minkowsky) space-time contribution, whilst the other addresses to the thermal and the topological effect. In such a case, a renormalized energy-momentum tensor is introduced in a consistent and simple way [26]. For the Casimir effect, it is convenient to work with the real-time canonical formalism.

We have organized this paper in the following way: in Section II, some aspects of TFD are presented to describe a field in a topology $S^1 \times S^1 \times R^3$. In Section III, the energy-momentum tensor of our model is derived. In Section IV, the topology $S^1 \times S^1 \times R^3$ is considered; and Section V, the Casimir effect for our model is studied. Concluding remarks are presented in Section VI.

II. THERMOFIELD DYNAMICS AND TOPOLOGY $S^1 \times S^1 \times R^3$

In accordance with the ref. [27] we present elements of thermofield dynamics(TFD), emphasizing aspects to be used in the calculation of the Casimir effect for our model. In short, TFD is introduced by two basic ingredients [26]. Considering a von-Neumann algebra

of operator in Hilbert space, there is a doubling, corresponding to the commutants introduced by a modular conjugation. This corresponds to a doubling of the original Fock space of the system leading to the expanded space $\mathcal{H}_T = \mathcal{H} \otimes \tilde{\mathcal{H}}$. This doubling is defined by a mapping $\tilde{\cdot}: \mathcal{H} \rightarrow \tilde{\mathcal{H}}$, associating each operator a acting on \mathcal{H} with two operators in \mathcal{H}_T , A and \tilde{A} , which are connected by the modular conjugation in a c^* -algebra, also called tilde conjugation rules [28, 29]:

$$\begin{aligned}(A_i A_j)^\sim &= \tilde{A}_i \tilde{A}_j, \\ (cA_i + A_j)^\sim &= c^* \tilde{A}_i + \tilde{A}_j, \\ (A_i^\dagger)^\sim &= (\tilde{A}_i)^\dagger, (\tilde{A}_i)^\sim = -\xi A_i,\end{aligned}$$

with $\xi = -1$ for bosons and $\xi = +1$ for fermions. The physical variables are described by nontilde operators. The tilde variables, defined in the commutant of the von Neumann algebra, are associated with generators of the modular group given by $\hat{A} = A - \tilde{A}$. With this elements, reducible representations of Lie-groups can be studied, in particular, kinematical symmetries as the Lorentz group. This gives rise to Liouville-von-Neumann-like equations of motion. The other basic ingredient of TFD is a Bogoliubov transformation, $B(\alpha)$, introducing a rotation in the tilde and non-tilde variables, such that thermal effects emerge from a condensate state. The rotation parameter α is associated with temperature, and this procedure is equivalent to the usual statistical thermal average. In the standard doublet notation [?], we write

$$(A^r(\alpha)) = \begin{pmatrix} A(\alpha) \\ \xi \tilde{A}^\dagger(\alpha) \end{pmatrix} = B(\alpha) \begin{pmatrix} A \\ \xi \tilde{A}^\dagger \end{pmatrix}, \quad (1)$$

$(A^r(\alpha))^\dagger = (A^\dagger(\alpha), \tilde{A}(\alpha))$, with the Bogoliubov transformation given by

$$B(\alpha) = \begin{pmatrix} u(\alpha) & -v(\alpha) \\ \xi v(\alpha) & u(\alpha) \end{pmatrix}, \quad (2)$$

where $u^2(\alpha) + \xi v^2(\alpha) = 1$.

The parametrization of the Bogoliubov transformation in TFD is obtained by setting $\alpha = \beta = T^{-1}$ and by requiring that the thermal average of the number operator, $N(\alpha) = a^\dagger(\alpha)a(\alpha)$, i.e. $\langle N(\alpha) \rangle_\alpha = \langle 0, \tilde{0} | a^\dagger(\alpha)a(\alpha) | 0, \tilde{0} \rangle$, gives either the Bose or the Fermi distribution, i.e

$$N(\alpha) = v^2(\beta) = (e^{\beta\varepsilon} + \xi)^{-1}. \quad (3)$$

Here we have used, for the sake of simplicity of notation, $A \equiv a$ and $\tilde{A} \equiv \tilde{a}$, and

$$a = u(\alpha)a(\alpha) + v(\alpha)\tilde{a}^\dagger(k, \alpha),$$

such that the other operators ($a^\dagger(k), \tilde{a}(k), \tilde{a}^\dagger(k)$) can be obtained by applying the Hermitian or the tilde conjugation rules. It is shown then that the thermal average, $\langle N(\alpha) \rangle_\alpha$, can be written as $\langle N(\alpha) \rangle_\alpha = \langle 0(\alpha) | a^\dagger a | 0(\alpha) \rangle$, where $|0(\alpha)\rangle$ is given by $|0(\alpha)\rangle = U(\alpha)|0, \tilde{0}\rangle$, with

$$U(\alpha) = \exp\{\theta(\alpha)[a^\dagger \tilde{a}^\dagger - a\tilde{a}]\}.$$

Let us consider the free Klein-Gordon field described by the Hamiltonian $\mathcal{H} = \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi - \frac{1}{2}m^2 \phi^2$, in a Minkowski space specified by the diagonal metric with signature $(+, - - -)$. The generalization of $U(\alpha)$ is then defined for all modes, such that

$$\begin{aligned}\phi(x; \alpha) &= U(\alpha)\phi(x)U^{-1}(\alpha), \\ \tilde{\phi}(x; \alpha) &= U(\alpha)\tilde{\phi}(x)U^{-1}(\alpha).\end{aligned}$$

Using a Bogoliubov transformation for each mode, we get [26]

$$\phi(x; \alpha) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k_0} [a(k; \alpha)e^{-ikx} + a^\dagger(k; \alpha)e^{ikx}]$$

and

$$\tilde{\phi}(x; \alpha) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k_0} [\tilde{a}(k; \alpha)e^{ikx} + \tilde{a}^\dagger(k; \alpha)e^{-ikx}].$$

The α -propagator is defined by

$$\begin{aligned}G(x - y, \alpha) &= -i\langle 0, \tilde{0} | T[\phi(x; \alpha)\phi(y; \alpha)] | 0, \tilde{0} \rangle \\ &= -i\langle 0(\alpha) | T[\phi(x)\phi(y)] | 0(\alpha) \rangle,\end{aligned}\tag{4}$$

where T is the time-ordering operator. This leads to

$$G_0(x - y, \alpha) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} G_0(k, \alpha),\tag{5}$$

where

$$G_0(k; \alpha) = G_0(k) + v^2(k; \alpha)[G_0(k) - G_0^*(k)],\tag{6}$$

with

$$G_0(k) = \frac{1}{k^2 - m^2 + i\varepsilon},$$

such that

$$G_0(k) - G_0^*(k) = 2\pi i \delta(k^2 - m^2).$$

Using $v^2(k_\alpha; \alpha) = v^2(k^0; \beta)$ as the boson distribution, $n(k^0; \beta)$, i.e.

$$v^2(k^0; \beta) = n(k^0; \beta) = \frac{1}{(e^{\beta\omega_k} - 1)} = \sum_{l_0=1}^{\infty} e^{-\beta k^0 l_0}, \quad (7)$$

with $\omega_k = k_0$ and $\beta = 1/T$, T being the temperature, then we have

$$G(k, \beta) = G_0(k) + 2\pi i n(k^0, \beta) \delta(k^2 - m^2), \quad (8)$$

with

$$G_0(x - y) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-y)} G_0(k).$$

For the case $m = 0$, we have

$$G_0(x - y) = \frac{-i}{(2\pi)^2} \frac{1}{(x - x')^2 - i\varepsilon}, \quad (9)$$

The Green function given in Eq. (4) is also written as

$$\begin{aligned} G_0(x - y, \beta) &= \text{Tr}[\rho(\beta) T[\phi(x)\phi(y)]] \\ &= G_0(x - y - i\beta n_0, \beta), \end{aligned}$$

where $\rho(\beta)$ is the equilibrium density matrix for the grand-canonical ensemble and $n_0 = (1, 0, 0, 0)$. This shows that $G_0(x - y, \beta)$ is a periodic function in the imaginary time, with period of β ; and the quantities $w_n = 2\pi n/\beta$ are the Matsubara frequencies. This periodicity is known as the KMS (Kubo, Martin, Schwinger) boundary condition. From Eq. (5) we show that $G_0(x - y, \beta)$ is a solution of the Klein-Gordon equation: with $\tau = it$, such that $\square + m^2 = -\partial_\tau^2 - \nabla^2 + m^2$, and

$$(\square + m^2)G_0(x, \beta) = -\delta(x). \quad (10)$$

Then $G_0(x - y, \beta)$ can also be written as a Fourier series, i.e.

$$G_0(x - y, \beta) = \frac{-1}{i\beta} \sum_n \int d^3 p \frac{e^{-ik_n \cdot x}}{k_n^2 - m^2 + i\varepsilon}, \quad (11)$$

where $k_n = (k_n^0, \mathbf{k})$. The propagator, given in Eq. (4) and in Eq. (11), is solution of Eq. (10) and fullfils the same boundary condition of periodicity and Feynman contour. Then these

solutions are the same. A direct proof is provided by Dolan and Jackiw in the case of temperature [30].

Due to the periodicity and the fact $G_0(x, \beta)$ and $G_0(x-y)$ satisfy Eq. (10), the same local structure, then this finite temperature theory results to be the $T = 0$ theory compactified in a topology $\Gamma_4^1 = S^1 \times \mathbb{R}^3$, where the (imaginary) time is compactified in S^1 , with circumference β . The Bogoliubov transformation introduces the imaginary compactification through a condensate.

For an Euclidian theory, this procedure can be developed for space compactification. In accordance with [27] the Bogoliubov transformation is given by

$$v^2(k^1, L_1) = \sum_{n=1}^{\infty} e^{-inL_1k^1}. \quad (12)$$

From this result, we compactify this theory in the imaginary time in order to take into account the temperature effect. We consider now the topology $\Gamma_4^2 = S^1 \times S^1 \times R^2$. The boson field is compactified in two directions, i.e. x^0 and x^1 . In the x^1 -axis, the compactification is in a circle of circumference L_1 and in the Euclidian x^0 -axis, the compactification is in a circumference β , such that in both of the cases the Green function satisfies periodic boundary conditions. In this case, the Bogoliubov transformation is given by

$$\begin{aligned} v^2(k^0, k^1; \beta, L_1) &= v^2(k^0; \beta) + v^2(k^1; L_1) \\ &+ 2v^2(k^0; \beta)v^2(k^1; L_1). \end{aligned} \quad (13)$$

This corresponds to a generalization of the Dolan-Jackiw propagator, describing a system of free bosons at finite temperature, with a compactified space dimension [26, 30]. Observe the following consistency relations

$$\begin{aligned} v_B^2(k^0; \beta) &= \lim_{L_1 \rightarrow \infty} v^2(k^0, k^1; \beta, L_1), \\ v_B^2(k^1; L_1) &= \lim_{\beta \rightarrow \infty} v^2(k^0, k^1; \beta, L_1). \end{aligned}$$

In the next sections we use these results to analyze the energy-momentum tensor of our model.

III. THE ELECTROMAGNETISM WITH LORENTZ BREAKING EVEN OF SME SECTOR

Now we begin this section analyzing a modified electromagnetism by a CPT-even term of the SME. A way of investigating the effects of the violation of the Lorentz symmetry is to consider parameters associated with vector and tensor fields (background), such as $K_{\mu\nu\kappa\lambda}$. Based on Maxwell's electrodynamics, these background fields should be very small because any effect associated with them is expected to be in an energy scale that we have never accessed. Both vector and tensor fields are considered to be background fields since they permeate the whole spacetime and we have no access to its source, i.e., they are fixed vector and tensor fields that select a privileged direction in the spacetime, and thus break the isotropy. Thereby, based on Maxwell's electrodynamics, the Lagrangian of the gauge sector of the Standard Model Extension is given by [5]

We propose to carry out our analysis by starting off from the action

$$\Sigma = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} + \Sigma_k, \quad (14)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field strength tensor.

$$\Sigma_k = -\frac{1}{4} \int d^4x (\kappa^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}) \quad (15)$$

where

$$\mathcal{L}_\kappa = - \left(\frac{1}{4} \kappa^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right), \quad (16)$$

Then we deal with a Lorentz symmetry violating tensor $K_{\mu\nu\kappa\lambda}$ in such a way that the CPT symmetry is preserved. In recent years, it has been shown in Ref. [31] that a particular decomposition of the tensor $K_{\mu\nu\kappa\lambda}$ produces a modification on the equations of motion of the electromagnetic waves due to the presence of vacuum anisotropies, which gives rise to the modified Maxwell equations. As a consequence, the anisotropy can be a source of the electric field and, then, Gauss's law is modified. Besides, the Ampère-Maxwell law is modified by the presence of anisotropies and it has a particular interest in the analysis of vortices solutions since it generates the dependence of the vortex core size on the intensity of the anisotropy.

The “tensor” $K_{\mu\nu\kappa\lambda}$ is CPT-even, i. e., doesn't violate the CPT symmetry. Despite CPT violation implies violation of Lorentz invariance, the reverse is not true. The action above is

Lorentz-violating in the sense that the “tensor” $K_{\mu\nu\kappa\lambda}$ has a non-zero vacuum expectation value. That “tensor” presents the following symmetries:

$$K_{\mu\nu\kappa\lambda} = K_{[\mu\nu][\kappa\lambda]}, \quad K_{\mu\nu\kappa\lambda} = K_{\kappa\lambda\mu\nu}, \quad K_{\mu\nu}{}^{\mu\nu} = 0, \quad (17)$$

as usually appears in the literature, we can reduce the degrees of freedom take into account the ansatz :

$$K_{\mu\nu\kappa\lambda} = \frac{1}{2} (\eta_{\mu\kappa} \tilde{\kappa}_{\nu\lambda} - \eta_{\mu\lambda} \tilde{\kappa}_{\nu\kappa} + \eta_{\nu\lambda} \tilde{\kappa}_{\mu\kappa} - \eta_{\nu\kappa} \tilde{\kappa}_{\mu\lambda}), \quad (18)$$

$$\tilde{\kappa}_{\mu\nu} = \kappa (\xi_\mu \xi_\nu - \eta_{\mu\nu} \xi^\alpha \xi_\alpha / 4), \quad (19)$$

$$\kappa = \frac{4}{3} \tilde{\kappa}^{\mu\nu} \xi_\mu \xi_\nu, \quad (20)$$

Using the decomposition in the term Lorentz violating term we have,

$$\kappa^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 2\kappa (g^{\mu\rho} (\xi^\nu \xi^\sigma - g^{\nu\sigma} \xi^e \xi_e / 4)) F_{\mu\nu} F_{\rho\sigma} \quad (21)$$

Then joint the Maxwell with the CPT-even violating term, we have,

$$\Sigma_g = \int d^4x \left\{ -\frac{1}{4} \left(1 - \frac{\kappa}{2} \xi^e \xi_e \right) F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2} \xi^\nu \xi^\sigma F^\rho{}_\nu F_{\rho\sigma} \right\} \quad (22)$$

To calculate the Casimir effect for this model we need the expression of the moment tensor-energy. With this expression we can evaluate the effect of violating the background Casimir effect.

IV. THE ENERGY-MOMENTUM TENSOR

The complete expression of the energy-momentum tensor is given by:

$$\mathcal{T}_{\alpha\beta} = - \left(1 - \frac{\kappa}{2} \xi^2 \right) F^\rho{}_\beta F_{\rho\alpha} + g_{\alpha\beta} \frac{1}{4} \left(1 - \frac{\kappa}{2} \xi^2 \right) F_{\mu\nu} F^{\mu\nu} - \kappa \xi^\nu \xi^\sigma F_{\alpha\nu} F_{\beta\sigma} + g_{\alpha\beta} \frac{\kappa}{2} \xi^\nu \xi^\sigma F^\rho{}_\nu F_{\rho\sigma}, \quad (23)$$

and establishing the relationship $\mathcal{T}_{\alpha\beta} = \mathcal{T}_{\alpha\beta}^a + \mathcal{T}_{\alpha\beta}^b$, where,

$$\mathcal{T}_{\alpha\beta}^a = - \left(1 - \frac{\kappa}{2}\xi^2\right) F^{\rho}{}_{\beta} F_{\rho\alpha} + g_{\alpha\beta} \frac{1}{4} \left(1 - \frac{\kappa}{2}\xi^2\right) F_{\mu\nu} F^{\mu\nu}; \quad (24)$$

$$\mathcal{T}_{\alpha\beta}^b = -\kappa \xi^{\nu} \xi^{\sigma} F_{\alpha\nu} F_{\beta\sigma} + g_{\alpha\beta} \frac{\kappa}{2} \xi^{\nu} \xi^{\sigma} F^{\rho}{}_{\nu} F_{\sigma\rho}. \quad (25)$$

Observing that the gauge potential satisfies the equation,

$$\theta_{\mu\nu} A^{\nu} = \left(g_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\square} \right) A^{\nu}(x) = 0, \quad (26)$$

with the conjugated four momomentum $\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_{\mu})}$, i.e. $\pi^0 = 0$ $\pi^i = \partial^0 A^i - \partial^i A^0$, and the comutation relation obeys,

$$[A_i(\mathbf{x}, t), \pi_j(\mathbf{x}', t)] = i \left[\delta_{ij} - \frac{1}{\nabla^2} \partial_i \partial_j \right] \delta(\mathbf{x}, \mathbf{x}'), \quad i, j, k = 1, 2, 3. \quad (27)$$

Now we begin to work with $\mathcal{T}_{\alpha\beta}^b$,

$$\mathcal{T}_{\alpha\beta}^b = -\kappa \mathcal{F}^{\sigma}{}_{\alpha\nu}(\mathbf{x}) \mathcal{F}^{\nu}{}_{\beta\sigma}(\mathbf{x}') + g_{\alpha\beta} \frac{\kappa}{2} \mathcal{F}^{\sigma\rho}{}_{\nu}(\mathbf{x}) \mathcal{F}^{\nu}{}_{\rho\sigma}(\mathbf{x}') \quad (28)$$

where

$$\mathcal{F}^{\sigma}{}_{\alpha\nu}(\mathbf{x}) = \xi^{\sigma} F_{\alpha\nu}; \quad \Sigma^{\mu}{}_{\nu} = \xi^{\mu} \partial_{\nu}. \quad (29)$$

To compute de vacuum expectation value,

$$\mathcal{T}_{\alpha\beta}^b(x) = \lim_{x \rightarrow x'} \left[-\kappa \mathcal{F}^{\sigma}{}_{\alpha\nu}(\mathbf{x}) \mathcal{F}^{\nu}{}_{\beta\sigma}(\mathbf{x}') + g_{\alpha\beta} \frac{\kappa}{2} \mathcal{F}^{\sigma\rho}{}_{\nu}(\mathbf{x}) \mathcal{F}^{\nu}{}_{\rho\sigma}(\mathbf{x}') \right] \quad (30)$$

in another way,

$$\mathcal{T}_{\alpha\beta}^b(x) = \lim_{x \rightarrow x'} \left[-\kappa \mathfrak{F}^{\sigma}{}_{\alpha\nu},{}^{\nu}{}_{\beta\sigma}(\mathbf{x}, \mathbf{x}') + g_{\alpha\beta} \frac{\kappa}{2} \mathfrak{F}^{\sigma\rho}{}_{\nu},{}^{\nu}{}_{\rho\sigma}(\mathbf{x}, \mathbf{x}') \right] \quad (31)$$

such that,

$$\mathfrak{F}^{\sigma}{}_{\alpha\nu}(\mathbf{x}, \mathbf{x}') = T \left[-\kappa \mathcal{F}^{\sigma}{}_{\alpha\nu}(\mathbf{x}) \mathcal{F}^{\nu}{}_{\beta\sigma}(\mathbf{x}') \right] \quad (32)$$

where T is the time ordering operator,

$$\begin{aligned} \mathfrak{F}^{\sigma}{}_{\alpha\nu}(\mathbf{x}, \mathbf{x}') &= (\Sigma^{\sigma}{}_{\alpha} A_{\beta}(\mathbf{x}) - \Sigma^{\sigma}{}_{\beta} A_{\alpha}(\mathbf{x})) (\Sigma^{\nu}{}_{\beta} A_{\sigma}(\mathbf{x}') - \Sigma^{\nu}{}_{\alpha} A_{\beta}(\mathbf{x}')) \theta(\mathbf{x}_0 - \mathbf{x}_0') + \\ &+ (\Sigma^{\sigma}{}_{\beta} A_{\alpha}(\mathbf{x}') - \Sigma^{\sigma}{}_{\alpha} A_{\beta}(\mathbf{x}')) (\Sigma^{\sigma}{}_{\alpha} A_{\beta}(\mathbf{x}) - \Sigma^{\sigma}{}_{\beta} A_{\alpha}(\mathbf{x})) \theta(\mathbf{x}' - \mathbf{x}), \end{aligned} \quad (33)$$

which can be written in terms of Γ such that,

$$\Gamma^{\sigma}_{\alpha\nu, \nu\beta\sigma}{}^{\lambda\psi}(\mathbf{x}, \mathbf{x}') = (\delta_{\nu}^{\lambda}\Sigma^{\sigma}_{\alpha} - \delta_{\alpha}^{\lambda}\Sigma^{\sigma}_{\nu}) \left(\delta_{\sigma}^{\psi}\Sigma^{\nu}_{\beta} - \delta_{\beta}^{\psi}\Sigma^{\nu}_{\sigma} \right), \quad (34)$$

with

$$\begin{aligned} \mathfrak{F}^{\sigma}_{\alpha\nu, \nu\beta\sigma}{}^{\lambda\psi}(\mathbf{x}, \mathbf{x}') &= \Gamma^{\sigma}_{\alpha\nu, \nu\beta\sigma}{}^{\lambda\psi}(\mathbf{x}, \mathbf{x}') T[A_{\lambda}(\mathbf{x}), A_{\psi}(\mathbf{x}')] + \\ &\quad - n_{0\alpha} \delta(\mathbf{x}_0 - \mathbf{x}_0') I_{\nu, \nu\beta\sigma}{}^{\lambda\psi}(\mathbf{x}, \mathbf{x}') + n_{0\nu} \delta(\mathbf{x}_0 - \mathbf{x}_0') I_{\alpha, \nu\beta\sigma}{}^{\lambda\psi}(\mathbf{x}, \mathbf{x}'), \end{aligned} \quad (35)$$

unless of expression,

$$I_{\alpha, \nu\beta\sigma}{}^{\lambda\psi}(\mathbf{x}, \mathbf{x}') = [A_{\alpha}(\mathbf{x}), \mathcal{F}^{\nu\beta\sigma}{}^{\lambda\psi}(\mathbf{x}')], \quad (36)$$

given by,

$$\begin{aligned} I_{\alpha, \nu\beta\sigma}{}^{\lambda\psi}(\mathbf{x}, \mathbf{x}') &= \xi^{\nu}(\mathbf{x}') \{ [A_{\alpha}(\mathbf{x}), \partial_{\beta} A_{\sigma}(\mathbf{x}')] - [A_{\alpha}(\mathbf{x}), \partial_{\sigma} A_{\beta}(\mathbf{x}')] \} + \\ &\quad + i n_{0\sigma} \xi^{\nu}(\mathbf{x}') (g_{\alpha\beta} - \nabla^{-2} \partial_{\alpha} \partial_{\beta}) \delta(\tilde{\mathbf{x}} - \tilde{\mathbf{x}}') - i n_{0\beta} \xi^{\nu}(\mathbf{x}') (g_{\alpha\sigma} - \nabla^{-2} \partial_{\alpha} \partial_{\sigma}) \delta(\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'), \end{aligned} \quad (37)$$

The tensor $\mathcal{T}_{\alpha\beta}^b(x)$ can be written as,

$$\begin{aligned} \mathcal{T}_{\alpha\beta}^b(x) &= \lim_{x \rightarrow x'} \left[(U_{\alpha\beta}^b(x) - \mathcal{V}_{\alpha\beta}^b(x)) \left(g_{\lambda\psi} + \frac{\partial_{\lambda} \partial_{\psi}}{\square} \right) \frac{1}{\square} \right], \\ U_{\alpha\beta}^b(x) &= \delta_{\nu}^{\lambda} \Sigma^{\sigma}_{\alpha} \delta_{\sigma}^{\psi} \Sigma^{\nu}_{\beta} - \delta_{\nu}^{\lambda} \Sigma^{\sigma}_{\alpha} \delta_{\beta}^{\psi} \Sigma^{\nu}_{\sigma} - \delta_{\alpha}^{\lambda} \Sigma^{\sigma}_{\nu} \delta_{\sigma}^{\psi} \Sigma^{\nu}_{\beta} + \delta_{\alpha}^{\lambda} \Sigma^{\sigma}_{\nu} \delta_{\beta}^{\psi} \Sigma^{\nu}_{\sigma}, \\ \mathcal{V}_{\alpha\beta}^b(x) &= \frac{1}{2} g_{\alpha\beta} (\delta_{\nu}^{\lambda} \Sigma^{\sigma}_{\rho} \delta_{\sigma}^{\psi} \Sigma^{\nu\rho} - \delta_{\nu}^{\lambda} \Sigma^{\sigma}_{\rho} \delta^{\psi\rho} \Sigma^{\nu}_{\sigma} - \delta_{\nu}^{\lambda} \Sigma^{\sigma}_{\nu} \delta_{\sigma}^{\psi} \Sigma^{\nu\rho} + \delta_{\rho}^{\lambda} \Sigma^{\sigma}_{\nu} \delta^{\psi\rho} \Sigma^{\nu}_{\sigma}), \end{aligned} \quad (38)$$

that considering only the first term of the propagator, after some operations $\mathcal{T}_{\alpha\beta} = \mathcal{T}_{\alpha\beta}^a + \mathcal{T}_{\alpha\beta}^b$, where,

$$\langle \mathcal{T}_{\alpha\beta}(x) \rangle = \lim_{x \rightarrow x'} \left[\Gamma_{\alpha\beta}(\mathbf{x}, \mathbf{x}') G_0(\mathbf{x}, \mathbf{x}') + 2i \left(n_{0\alpha} n_{0\beta} - \frac{1}{2} g_{\alpha\beta} \delta(\tilde{\mathbf{x}} - \tilde{\mathbf{x}}') \right) \right]; \quad (39)$$

$$\rangle \mathcal{D}_{\alpha\beta} = \langle 0 | T[A_{\alpha}(\mathbf{x}) A_{\beta}(\mathbf{x}')] | 0 \rangle = g_{\alpha\beta} G_0(\mathbf{x}, \mathbf{x}') \quad (40)$$

with

$$G_0(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi^2 i} \frac{1}{(\mathbf{x} - \mathbf{x}')}, \quad (41)$$

$$\rangle \mathcal{D}_{\alpha\beta} = \langle 0 | T[A_{\alpha}(\mathbf{x}) A_{\beta}(\mathbf{x}')] | 0 \rangle = g_{\alpha\beta} G_0(\mathbf{x}, \mathbf{x}'). \quad (42)$$

For dependent fields in a parameter ϵ , the renormalized energy-momentum tensor is:

$$\langle \mathcal{T}_{\alpha\beta}(x; \epsilon) \rangle = -i \lim_{x \rightarrow x'} \left[\Gamma_{\alpha\beta}(\mathbf{x}, \mathbf{x}') G_0(\mathbf{x}, \mathbf{x}'; \epsilon) + 2i \left(n_{0\alpha} n_{0\beta} - \frac{1}{2} g_{\alpha\beta} \delta(\tilde{\mathbf{x}} - \tilde{\mathbf{x}}') \right) \right]; \quad (43)$$

or in another way,

$$\langle \mathcal{T}_{\alpha\beta}(x; \epsilon) \rangle = \langle \mathcal{T}_{\alpha\beta}(x; \epsilon) \rangle - \langle \mathcal{T}_{\alpha\beta}(x) \rangle; \quad (44)$$

$$\langle \mathcal{T}_{\alpha\beta}(x; \epsilon) \rangle = -i \lim_{x \rightarrow x'} [\Gamma_{\alpha\beta}(\mathbf{x}, \mathbf{x}') G_0(\mathbf{x}, \mathbf{x}'; \epsilon)] \quad (45)$$

which can be written in terms of Γ

$$\Gamma_{\alpha\beta}(\mathbf{x}, \mathbf{x}') = 2\xi_\nu \xi^\nu \left(\partial_\alpha \partial_\beta - \frac{1}{2} g_{\alpha\beta} \partial_\alpha \partial^\beta \right) - 2(\xi^\sigma \partial_\sigma \partial_\alpha \xi_\beta), \quad (46)$$

V. THE ELECTROMAGNETISM WITH LORENTZ BREAKING EVEN OF SME SECTOR α -ENERGY-MOMENTUM TENSOR

In this section we calculate energy-momentum for our model in a compactified in a toroidal topology. We define the physical (renormalized) energy-momentum tensor by

$$\mathcal{T}^{\mu\nu}(x; \alpha) = \langle T^{\mu\nu}(x; \alpha) \rangle_0 - \langle T^{\mu\nu}(x) \rangle_0 \quad (47)$$

where $\langle T^{\mu\nu}(x; \alpha) \rangle_0 = \langle 0 | T^{\mu\nu}(x, \alpha) | 0 \rangle \equiv \langle \alpha | T^{\mu\nu}(x) | \alpha \rangle$. This leads to

$$\mathcal{T}^{\mu\nu}(x; \alpha) = -i \lim_{x \rightarrow x'} \Gamma^{\mu\nu}(x, x') G_0(x - x'; \alpha),$$

where

$$\begin{aligned} \overline{G}(x - x'; \alpha) &= G_0(x - x'; \alpha) - G_0(x - x') \\ &= \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-x')} v^2(k_\alpha, \alpha) [G_0(k) - G_0^*(k)]. \end{aligned}$$

Let us calculate, as an example, the case of temperature defined by $\alpha = (\beta, 0, 0, 0)$, with $v^2(k_0; \beta)$ given by Eq. (7). Then we have

$$\mathcal{T}_{\alpha\beta}(\beta) = -\frac{\kappa\pi^2}{90n_0^4} \left\{ \xi^2 \left(\frac{8n_{0\alpha}n_{0\beta}}{n_0^2} + 5g_{\alpha\beta} \right) \right\}, \quad (48)$$

taking into account the $\mathcal{T}_{\alpha\beta}^a$ term and obtaining its contribution we have the complete expression of the energy-momentum tensor,

$$\begin{aligned} \mathcal{T}_{\alpha\beta}(\beta) = & -\frac{\pi^2}{45\beta^4} \left(1 - \frac{\kappa}{2}\xi^2\right) (g_{\alpha\beta} - 4n_{0\alpha}n_{0\beta}) + \\ & -\frac{\kappa\pi^2}{90n_0^4\beta^4} \left\{ \xi^2 \left(\frac{8n_{0\alpha}n_{0\beta}}{n_0^2} + 5g_{\alpha\beta} \right) + \frac{8}{n_0^2} (\xi^\sigma n_{0\sigma}n_{0\alpha}\xi_\beta + \xi^\sigma n_{0\sigma}\xi_\alpha n_{0\beta}) - 4\xi_\alpha\xi_\beta \right\} \end{aligned} \quad (49)$$

where $n_0^\mu = (1, 0, 0, 0)$. Using the Riemann Zeta function

$$\zeta(4) = \sum_{l=1}^{\infty} \frac{1}{l^4} = \frac{\pi^4}{90},$$

we obtain

$$\mathcal{T}_{00}(\beta) = \frac{3\pi^2}{45\beta^4} \left(1 - \frac{\kappa}{2}\xi^2\right) - \frac{\kappa\pi^2}{90\beta^4} \left\{ 13\xi^2 + \frac{16\xi^0\xi_0}{n_0^2} - 4\xi_0\xi_0 \right\}, \quad (50)$$

This leads to the Stephan-Boltzmann law for our model, since the energy and pressure are given respectively by,

$$E(\beta) = \mathcal{T}_{00}(\beta) = \begin{cases} \left(1 - \frac{\kappa}{2}\right) \frac{\pi^2}{15\beta^4} - 25 \frac{\kappa\pi^2}{90\beta^4}, & \text{for } \xi_\rho \text{ time-like} \\ \left(1 + \frac{\kappa}{2}\right) \frac{\pi^2}{15\beta^4} - 13 \frac{\kappa\pi^2}{90\beta^4}, & \text{for } \xi_\rho \text{ space-like} \end{cases} \quad (51)$$

and

$$P(\beta) = \mathcal{T}^{33}(\beta) = \begin{cases} \left(1 - \frac{\kappa}{2}\right) \frac{\pi^2}{45\beta^4} + \frac{\kappa\pi^2}{10\beta^4}, & \text{for } \xi_\rho \text{ time-like} \\ \left(1 + \frac{\kappa}{2}\right) \frac{\pi^2}{45\beta^4} - \frac{\kappa\pi^2}{90\beta^4}, & \text{for } \xi_\rho \text{ space-like} \end{cases} \quad (52)$$

In the next section, we use a similar procedure to calculate the Casimir effect.

VI. THE CASIMIR EFFECT FOR THE MODEL

Initially we consider the Casimir effect at zero temperature. This is given by our energy-momentum tensor $\mathcal{T}^{\mu\nu}(x; \alpha)$ given in Eq. (47), where α accounts for spatial compactifications. We take $\alpha = (0, 0, 0, iL)$, with L being the circumference of S^1 . The Bogoliubov transformation is given in Eq. (12), that in the present notation reads

$$v^2(k_3; L) = \sum_{l_3=1}^{\infty} e^{-iLk^3l_3}.$$

Thus $\mathcal{T}^{\mu\nu}(x; L)$ is given by

$$\begin{aligned}\mathcal{T}_{\alpha\beta}(L) &= -\frac{\pi^2}{45(n_3L)^4} \left(1 - \frac{\kappa}{2}\xi^2\right) (g_{\alpha\beta} + 4n_{3\alpha}n_{3\beta}) - \frac{\kappa\pi^2}{90(n_3L)^4} X_{\alpha\beta}(L) \\ X_{\alpha\beta}(L) &= \xi^2 \left(\frac{8n_{3\alpha}n_{3\beta}}{(n_3)^2} + 5g_{\alpha\beta}\right) + \frac{8}{(n_3)^2} (\xi^\sigma n_{3\sigma} n_{3\alpha} \xi_\beta + \xi^\sigma n_{3\sigma} \xi_\alpha n_{3\beta}) - 4\xi_\alpha \xi_\beta,\end{aligned}\quad (53)$$

where $n_{3\alpha} = (0, 0, 0, 1)$.

For the electromagnetic field with Lorentz breaking symmetry, the Casimir effect is calculated for plates apart from each other by a distance a , that is related to L [26]. We consider this fact for the sake of comparison. The Casimir energy

$$E(\beta) = \mathcal{T}_{00}(\beta) = \begin{cases} -\left(1 - \frac{\kappa}{2}\right) \frac{\pi^2}{45} \frac{1}{L^4} - \frac{\kappa\pi^2}{90} \frac{1}{L^4}, & \text{for } \xi_\rho \text{ time-like} \\ -\left(1 + \frac{\kappa}{2}\right) \frac{\pi^2}{45} \frac{1}{L^4} + \frac{\kappa\pi^2}{10} \frac{1}{L^4}, & \text{for } \xi_\rho \text{ space-like} \end{cases} \quad (54)$$

and pressure, respectively, are then given by

$$P(\beta) = \mathcal{T}^{33}(\beta) = \begin{cases} -\left(1 - \frac{\kappa}{2}\right) \frac{\pi^2}{15} \frac{1}{L^4} - \frac{17\kappa\pi^2}{90L^4} & \text{for } \xi_\rho \text{ time-like} \\ -\left(1 + \frac{\kappa}{2}\right) \frac{\pi^2}{15} \frac{1}{L^4} - \frac{23\kappa\pi^2}{90} \frac{1}{L^4}, & \text{for } \xi_\rho \text{ space-like} \end{cases} \quad (55)$$

It is interesting to compare such a result with the Casimir effect for the electromagnetic field. For the electromagnetic field with Lorentz breaking symmetry, the Casimir energy, at $T = 0$ K and for ξ_ρ time-like, it is exactly the same for the electromagnetic field without the extended term, eqs 14-16, $E(T) = -\pi^2/720a^4$, where $L = 2a$, while for ξ_ρ space-like, this result is to increase of, $E(T) = -4\kappa\pi^2/720a^4$. Already for Casimir pressure, as for ξ_ρ time-like as well for space-like, the extended term contributes to the final result.

The effect of temperature is introduced by taken $\alpha = (i\beta, 0, 0, iL)$. Using Eq. (13), $v^2(k^0, k^3; \beta, L)$ is given by

$$\begin{aligned}v^2(k^0, k^3; \beta, L) &= v^2(k^0; \beta) + v^2(k^3; L) + 2v^2(k^0; \beta)v^2(k^3; L) \\ &= \sum_{l_0=1}^{\infty} e^{-\beta k^0 l_0} + \sum_{l_3=1}^{\infty} e^{-iL k^3 l_3} + 2 \sum_{l_0, l_3=1}^{\infty} e^{-\beta k^0 l_0 - iL k^3 l_3}.\end{aligned}\quad (56)$$

The two parts of the energy-momentum tensor is $\mathcal{T}_{\alpha\beta}^A(\beta, L)$ and $\mathcal{T}_{\alpha\beta}^B(\beta, L)$,

$$\begin{aligned}\mathcal{T}_{\alpha\beta}^A(\beta, L) &= -\frac{2}{\pi^2} \left(1 - \frac{\kappa}{2}\xi^2\right) \left\{ \sum_{l_0=1}^{\infty} \mathcal{X}_{\alpha\beta}^A(\beta, L) + \sum_{l_3=1}^{\infty} \mathcal{Y}_{\alpha\beta}^A(\beta, L) + 4 \sum_{l_0, l_3=1}^{\infty} \mathcal{Z}_{\alpha\beta}^A(\beta, L) \right\}, \\ \mathcal{X}_{\alpha\beta}^A(\beta, L) &= \frac{g_{\alpha\beta} - 4n_{0\alpha}n_{0\beta}}{(\beta l_0)^4}, \quad \mathcal{Y}_{\alpha\beta}^A(\beta, L) = \frac{g_{\alpha\beta} - 4n_{0\alpha}n_{0\beta}}{(Ll_3)^4}, \\ \mathcal{Z}_{\alpha\beta}^A(\beta, L) &= \frac{(\beta l_0)^2 [g_{\alpha\beta} - 4n_{0\alpha}n_{0\beta}] + (2Ll_3)^2 [g_{\alpha\beta} + 4n_{3\alpha}n_{3\beta}]}{[(\beta l_0)^2 + (Ll_3)^2]^3},\end{aligned}\tag{57}$$

and

$$\begin{aligned}\mathcal{T}_{\alpha\beta}^B(\beta, L) &= \frac{-2K}{\pi^2} \sum_{l_0, l_3=1}^{\infty} \xi^2 \left(\frac{-8 [(Ll_3)^2 n_{3\alpha}n_{3\beta} - (\beta l_0)^2 n_{0\alpha}n_{0\beta}]}{[(Ll_3)^2 - (\beta l_0)^2]^3} + \frac{5g_{\alpha\beta} [(Ll_3)^2 - (\beta l_0)^2]}{[(Ll_3)^2 - (\beta l_0)^2]^3} \right) + \\ &+ 16 \frac{[(Ll_3)^2 \xi^\sigma n_{3\sigma} n_{3\alpha} \xi_\beta - (\beta l_0)^2 \xi^\sigma n_{0\sigma} \xi_\beta n_{0\alpha}]}{[(Ll_3)^2 - (\beta l_0)^2]^3} - \frac{4\xi_\alpha \xi_\beta}{[(Ll_3)^2 - (\beta l_0)^2]}\end{aligned}\tag{58}$$

Then taking into account the total expression of the energy-momentum tensor is $\mathcal{T}_{\alpha\beta}(\beta, L) = \mathcal{T}_{\alpha\beta}^A(\beta, L) + \mathcal{T}_{\alpha\beta}^B(\beta, L)$, we evaluate the Casimir energy $E(\beta, L)$ and the Casimir pressure $P(\beta, L)$ with the temperature dependence. The Casimir energy $\mathcal{T}^{00}(\beta)$ and pressure $\mathcal{T}^{33}(\beta)$ are given in the case of $\xi^\mu = (1; 0, 0, 0)$, respectively by

$$\begin{aligned}E(\beta, L) &= \left(1 - \frac{\kappa}{2}\right) \left(A(\beta, L) + \frac{8}{\pi^2} \sum_{l_0, l_3=1}^{\infty} B(\beta, L) \right) + \left(1 - \frac{\kappa}{2}\right) \left(\frac{2K}{\pi^2} \sum_{l_0, l_3=1}^{\infty} C(\beta, L) \right), \\ A(\beta, L) &= \frac{\pi^2}{45} \left(\frac{3}{\beta^4} - \frac{1}{L^4} \right), \\ B(\beta, L) &= \frac{3(\beta l_0)^2 - 4(Ll_3)^2}{[(\beta l_0)^2 + (Ll_3)^2]^3}, \\ C(\beta, L) &= \frac{8(\beta l_0)^2}{[(Ll_3)^2 - (\beta l_0)^2]^3} - \frac{5}{[(Ll_3)^2 - (\beta l_0)^2]^2} + \frac{4}{[(Ll_3)^2 - (\beta l_0)^2]},\end{aligned}\tag{59}$$

and

$$\begin{aligned}P(\beta, L) &= \left(1 - \frac{\kappa}{2}\right) \left(D(\beta, L) + \frac{8}{\pi^2} \sum_{l_0, l_3=1}^{\infty} F(\beta, L) + \frac{2K}{\pi^2} \sum_{l_0, l_3=1}^{\infty} G(\beta, L) \right), \\ D(\beta, L) &= \frac{\pi^2}{45} \left(\frac{1}{\beta^4} - \frac{3}{L^4} \right), \\ F(\beta, L) &= \frac{3(\beta l_0)^2 - 12(Ll_3)^2}{[(\beta l_0)^2 + (Ll_3)^2]^3}, \\ G(\beta, L) &= \frac{8(Ll_3)^2}{[(Ll_3)^2 - (\beta l_0)^2]^3} + \frac{5}{[(Ll_3)^2 - (\beta l_0)^2]^2}.\end{aligned}\tag{60}$$

The Casimir energy and pressure are given in the case of $\xi^\mu = (0; 0, 0, 1)$, respectively by

$$\begin{aligned}
E(\beta, L) &= \left(1 + \frac{\kappa}{2}\right) \left(H(\beta, L) + \frac{8}{\pi^2} \sum_{l_0, l_3=1}^{\infty} I(\beta, L) + \frac{2K}{\pi^2} \sum_{l_0, l_3=1}^{\infty} J(\beta, L) \right), \\
H(\beta, L) &= \frac{\pi^2}{45} \left(\frac{3}{\beta^4} - \frac{1}{L^4} \right), \\
I(\beta, L) &= \frac{3(\beta l_0)^2 - 4(Ll_3)^2}{[(\beta l_0)^2 + (Ll_3)^2]^3}, \\
J(\beta, L) &= \frac{-8(\beta l_0)^2}{[(Ll_3)^2 - (\beta l_0)^2]^3} + \frac{5}{[(Ll_3)^2 - (\beta l_0)^2]^2},
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
P(\beta, L) &= \left(1 + \frac{\kappa}{2}\right) \left(L(\beta, L) + \frac{8}{\pi^2} \sum_{l_0, l_3=1}^{\infty} M(\beta, L) + \frac{2K}{\pi^2} \sum_{l_0, l_3=1}^{\infty} N(\beta, L) \right), \\
L(\beta, L) &= \frac{\pi^2}{45} \left(\frac{1}{\beta^4} - \frac{3}{L^4} \right), \\
M(\beta, L) &= \frac{3(\beta l_0)^2 - 12(Ll_3)^2}{[(\beta l_0)^2 + (Ll_3)^2]^3}, \\
N(\beta, L) &= \frac{8(Ll_3)^2}{[(Ll_3)^2 - (\beta l_0)^2]^3} - \frac{5}{[(Ll_3)^2 - (\beta l_0)^2]^2} - \frac{4}{[(Ll_3)^2 - (\beta l_0)^2]}.
\end{aligned} \tag{62}$$

The first two terms of these expressions are, respectively, the Stephan-Boltzmann term and the Casimir effect at $T = 0$. The last term accounts for the simultaneous effect of spatial compactification, described by L , and temperature, $T = 1/\beta$.

VII. CONCLUDING REMARKS

In this work, we have adopted the Thermo-Field Dynamics (TFD) approach, a real-time formalism for Quantum Field Theory at finite temperatures, to study the Casimir Effect in the framework of an electrodynamical model with the so-called even Lorentz-symmetry breaking term in the photon sector of the SME. We have initially worked out the expression for the energy-momentum tensor of the model in terms of the TFD propagator, by considering an $S^1 \times S^1 \times R^2$ -topology, where the two factors S^1 correspond to a compactified space-like coordinate and the finite temperature. The TFD techanalities are very appropriated to deal with the renormalized energy-momentum tensor of the model with LSV under consideration. The Casimir energy and pressure have been both explicitly calculated and we find that the

corrections that arise directly from the LSV come out linear in the κ -parameter, which is constrained by several tests. The expressions we calculate for the energy and pressure can be used for the attainment of a new category of bound on the κ -parameter. Different cases have been considered where the external vector responsible for the LSV may be time- and space-like. Though not explicitly mentioned, the light-like case also gives rise to non-trivial effects on the energy and pressure. In the case in which the external vector responsible for the LSV is time-like the Casimir effect do not present influence of the Lorentz breaking.

Clearly, the tests require extremely high-precision measurements, once the known limits on κ are very tiny. However, for very high temperatures, the combined effect between the LSV parameter and the temperature itself may yield a measurable effect on the energy and pressure. On the other hand, one might adopt current measurements of the energy and pressure at finite temperatures to set up a new class of limits on the κ -parameter. A point which remains to be investigated in the LSV scenario we are considering is the thermal Casimir effect in the interaction of graphene with a metal. In this type of system we could enhance the effect of LSV and we could end up with more stringent constraints on the LSV parameters.

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